J. Math. Anal. Appl. 360 (2009) 398–411

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global existence of classical solutions to the mixed initial–boundary value problem for quasilinear hyperbolic systems of diagonal form with large BV data

Zhi-Qiang Shao [1]

Department of Mathematics, Fuzhou University, Fuzhou 350002, China

ARTICLE INFO

Article history:
Received 7 February 2009
Available online 26 June 2009
Submitted by R. Manásevich

Keywords:
Quasilinear hyperbolic system of diagonal form
Mixed initial–boundary value problem
Global classical solution
Linear degeneracy
Large BV data

ABSTRACT

In this paper, we investigate the mixed initial–boundary value problem with large BV data for linearly degenerate quasilinear hyperbolic systems of diagonal form with general nonlinear boundary conditions in the half space $\{(t,x) \mid t \geqslant 0,\ x \geqslant 0\}$. As the result in [A. Bressan, Contractive metrics for nonlinear hyperbolic systems, Indiana Univ. Math. J. 37 (1988) 409–421] suggests that one may achieve global smoothness even if the C^1 norm of the initial data is large, we prove that, if the C^1 norm and the BV norm of the initial and boundary data are bounded but possibly large, then the solution remains C^1 globally in time and possesses uniformly bounded total variation in x for all $t \geqslant 0$. As an application, we apply the result to the system describing the motion of relativistic closed strings in the Minkowski space R^{1+n}.

1. Introduction and main result

Consider the following first order quasilinear strictly hyperbolic system of diagonal form

$$\frac{\partial u_i}{\partial t} + \lambda_i(u)\frac{\partial u_i}{\partial x} = 0 \quad (i = 1, \dots, n), \tag{1.1}$$

where $u = (u_1, \dots, u_n)^T$ is the unknown vector-valued function of (t,x), $\lambda_i(u)$ $(i = 1, \dots, n)$ are assumed to be the C^2 vector-valued functions of u and are linearly degenerate, i.e.,

$$\frac{\partial \lambda_i(u)}{\partial u_i} \equiv 0 \quad (i = 1, \dots, n) \tag{1.2}$$

for any given u on the domain under consideration.

It is assumed that system (1.1) is strictly hyperbolic on the domain under consideration, i.e., the eigenvalues satisfy

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.3}$$

We assume that there exists a positive constant δ such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geqslant \delta \quad (i = 1, \dots, n-1). \tag{1.4}$$

E-mail address: zqshao_fzu@yahoo.com.cn.

[1] Supported by the National Natural Science Foundation of China (Grant No. 70371025), the Scientific Research Foundation of the Ministry of Education of China (Grant No. 02JA790014), the Natural Science Foundation of Fujian Province of China (Grant No. 2009J01006), and the Science and Technology Developmental Foundation of Fuzhou University (Grant No. 2004-XQ-16).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.06.066

We also assume that on the domain under consideration, the eigenvalues satisfy the non-characteristic condition

$$\lambda_j(u)+c<0<\lambda_k(u)-c \quad (j=1,\dots,m;\ k=m+1,\dots,n), \tag{1.5}$$

where c is a suitable positive constant.

Consider the mixed initial–boundary value problem for system (1.1) in the half space

$$D=\{(t,x)\mid t\geqslant 0,\ x\geqslant 0\} \tag{1.6}$$

with the initial condition

$$t=0:\quad u=\varphi(x) \quad (x\geqslant 0) \tag{1.7}$$

and the nonlinear boundary condition (cf. [11,12,19–21])

$$x=0:\quad u_s=g_s\big(\alpha(t),u_1,\dots,u_m\big)+h_s(t),\quad s=m+1,\dots,n\ (t\geqslant 0), \tag{1.8}$$

where

$$\alpha(t)=\big(\alpha_1(t),\dots,\alpha_k(t)\big).$$

Here, $g_s\in C^1 (s=m+1,\dots,n)$, $\varphi=(\varphi_1,\dots,\varphi_n)^T$, α and $h(\cdot)=(h_{m+1}(\cdot),\dots,h_n(\cdot))\in C^1$ with bounded C^1 norm, such that

$$\|\varphi(x)\|_{C^1},\ \|\alpha(t)\|_{C^1},\ \|h(t)\|_{C^1}\leqslant M, \tag{1.9}$$

for some positive constant M (bounded but possibly large). Also, we assume that the conditions of C^1 compatibility are satisfied at the point $(0,0)$. Without loss of generality, we assume that

$$g_s\big(\alpha(t),0,\dots,0\big)\equiv 0 \quad (s=m+1,\dots,n). \tag{1.10}$$

In this paper, we furthermore assume that the initial and boundary data are of bounded but possibly large total variation, i.e.,

$$TV(\varphi):=\int_0^{+\infty}|\varphi'(x)|\,dx\leqslant N,\qquad TV(\alpha):=\int_0^{+\infty}|\alpha'(t)|\,dt\leqslant N \tag{1.11}$$

and

$$TV(h):=\int_0^{+\infty}|h'(t)|\,dt\leqslant N, \tag{1.12}$$

for some positive constant N (bounded but possibly large).

Our goal in this paper is to prove the global existence of classical solutions to the mixed initial–boundary value problem (1.1) and (1.7)–(1.8) with large BV data. Our main result is the following:

Theorem 1.1 *(Global existence). Suppose that system* (1.1) *is strictly hyperbolic and linearly degenerate. Suppose furthermore that in a neighborhood of $u=0$, $\lambda_i(u)\in C^2$ $(i=1,\dots,n)$ and* (1.4)–(1.5) *hold. Suppose finally that φ, α, g_s, h_s $(s=m+1,\dots,n)$ are all C^1 functions with respect to their arguments satisfying the conditions of C^1 compatibility at the point* $(0,0)$. *If* (1.9) *and* (1.11)–(1.12) *hold for some positive constants M and N* (*bounded but possibly large*)*, then the mixed initial–boundary value problem* (1.1) *and* (1.7)–(1.8) *admits a unique global C^1 solution $u=u(t,x)$, defined in the half space $\{(t,x)\mid t\geqslant 0,\ x\geqslant 0\}$, with bounded total variation in x uniformly for all $t\geqslant 0$.*

Remark 1.1. Suppose that system (1.1) is non-strictly hyperbolic but each characteristic has a constant multiplicity, say, on the domain under consideration,

$$\lambda_1(u)<\cdots<\lambda_m(u)<0<\lambda_m(u)<\cdots<\lambda_{p+1}(u)\equiv\cdots\equiv\lambda_n(u) \quad (m\leqslant p\leqslant n). \tag{1.13}$$

Then the conclusion of Theorem 1.1 still holds (cf. [5]).

For the Cauchy problem of general quasilinear hyperbolic systems

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=0, \tag{1.14}$$

$$u(0,x)=\varphi(x), \tag{1.15}$$

such kinds of problems have been extensively studied (for instance, see [3–5,7,8,14,15,22] and references therein). In particular, Li et al. [14,15] proved that for any given initial data $\varphi(x)$ satisfying the following small and decaying property:

$$\sup_{x\in R}\big(1+|x|\big)^{1+\mu}\big(|\varphi(x)|+|\varphi'(x)|\big)\ll 1, \tag{1.16}$$

where $\mu > 0$ is a constant, the Cauchy problem (1.14), (1.15) admits a unique global C^1 solution with small C^1 norm, provided that system (1.14) is weakly linearly degenerate. In their works, the condition $\mu > 0$ is essential. If $\mu = 0$, a counterexample was constructed by Kong [8] showing that the C^1 solution may blow up in a finite time, even when system (1.14) is weakly linearly degenerate. However, it is well known that the BV space is a suitable framework for one-dimensional Cauchy problem for quasilinear hyperbolic systems (see Bressan [4]), the result in Bressan [3] suggests that one may achieve global smoothness even if the C^1 norm of the initial data is large. So the following question arises naturally: can we obtain the global existence of classical solutions of linearly degenerate quasilinear hyperbolic systems, provided that the BV norm of the initial data is suitably small? Here, it is important to mention that this problem was solved by Bressan [3], Zhou [22], Dai and Kong [5]. The mixed initial–boundary value problem case was also studied in Shao [19]. On the other hand, by the study of Amadori and Shen [1,2], J. Glimm and P.D. Lax [6], T. Nishida and J.A. Smoller [17], D. Serre [18] etc., we know that global existence of solutions to quasilinear hyperbolic systems with large BV data is a more difficult, still largely open problem. Therefore, it is natural to consider the following problem: can we obtain the global existence of classical solutions to the mixed initial–boundary value problem for linearly degenerate quasilinear hyperbolic systems, provided that the BV norm of the initial and boundary data is bounded but possibly large? Our work may provide a simpler approach to solve this problem, in this connection let us also mention Li and Peng's work [10,11], Liu and Zhou's work [16] on quasilinear hyperbolic systems in diagonal form.

The remainder of the paper is organized as follows. In Section 2 we first establish some uniform a priori estimates, these estimates will play an important role in the proof of main result. Then we prove the main result, Theorem 1.1. An application is discussed in Section 3. Concluding summary is given at the end of the paper, in Section 4.

Finally, we give some specific physical situations for which our work is of importance.

Example 1 *(Mixed initial–boundary value problem for the equation of time-like extremal surfaces in the Minkowski space–time $R^{1+(1+n)}$).* The extremal surfaces play an important role in the theoretical apparatus of elementary particle physics. A free string is a one-dimensional physical object whose motion is represented by a time-like extremal surfaces in the Minkowski space. Here, we are concerned with the global existence and uniqueness of classical solutions to the mixed initial–boundary value problem for the equation of time-like extremal surfaces in Minkowski space $R^{1+(1+n)}$ with large BV data. By $(x_0, x_1, \dots, x_{n+1})$ we denote a point in the $(1+(1+n))$-dimensional Minkowski space endorsed with the metric

$$ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2. \tag{1.17}$$

Let

$$x_0 = t, \quad x_1 = x, \quad x_2 = \phi_1(t,x), \quad \dots, \quad x_{n+1} = \phi_n(t,x) \tag{1.18}$$

be a two-dimensional surface. Then the induced metric on the surface is

$$d_*s^2 = -dt^2 + dx^2 + (d\phi_1)^2 + \cdots + (d\phi_n)^2 = -\big(1-(\phi_t)^2\big)\,dt^2 + \big(1+(\phi_x)^2\big)\,dx^2 + 2\phi_t\cdot\phi_x\,dx\,dt, \tag{1.19}$$

where $\phi = (\phi_1, \dots, \phi_n)^T$, ϕ_t or ϕ_x denote partial differentiation with respect to t or x respectively and $\cdot$ denotes inner product in R^n. We assume that the surface is time-like, i.e., the induced metric is Lorentzian. Thus, it is easy to see that the area of the surface is

$$I = \iint \sqrt{1 + |\phi_x|^2 - |\phi_t|^2 - |\phi_t|^2|\phi_x|^2 + (\phi_t\cdot\phi_x)^2}\,dx\,dt. \tag{1.20}$$

The surface is called an extremal surface if ϕ is the critical point of the area functional. The corresponding Euler–Lagrange equation is (cf. [5,23,24])

$$\left(\frac{\phi_t + |\phi_x|^2\phi_t - (\phi_t\cdot\phi_x)\phi_x}{\sqrt{1 + |\phi_x|^2 - |\phi_t|^2 - |\phi_t|^2|\phi_x|^2 + (\phi_t\cdot\phi_x)^2}}\right)_t - \left(\frac{\phi_x - |\phi_t|^2\phi_x + (\phi_t\cdot\phi_x)\phi_t}{\sqrt{1 + |\phi_x|^2 - |\phi_t|^2 - |\phi_t|^2|\phi_x|^2 + (\phi_t\cdot\phi_x)^2}}\right)_x = 0. \tag{1.21}$$

Eq. (1.21) is the equation of time-like extremal surfaces in the Minkowski space $R^{1+(1+n)}$. The extremal surfaces in the Minkowski space are C^2 surfaces with vanishing mean curvature. This is an interesting model in Lorentzian geometry. The Cauchy problem for the equation of time-like extremal surfaces was studied by Kong, Sun and Zhou [23], Dai and Kong [5]. They give the necessary and sufficient condition on the global existence of classical solutions of Eq. (1.21) with the initial data

$$\phi(0,x) = f(x), \qquad \phi_t(0,x) = g(x). \tag{1.22}$$

In this paper, we will consider the mixed initial–boundary value problem of Eq. (1.21) with the following data:

$$t = 0: \quad \phi = f(x), \quad \phi_t = g(x), \quad x \geqslant 0, \tag{1.23}$$

$$x = 0: \quad \phi_x = 0, \quad t \geqslant 0, \tag{1.24}$$

where $f'(x)$ and $g(x) \in C^1(R^+)$ with bounded C^1 norm. Also, we assume that the conditions of C^2 compatibility are satisfied at the point $(0,0)$.

Let

$$u = \phi_x, \qquad v = \phi_t, \tag{1.25}$$

where $u = (u_1, \dots, u_n)^T$ and $v = (v_1, \dots, v_n)^T$. Then (1.21) can be equivalently rewritten as

$$\begin{cases} u_t - v_x = 0, \\ \left(\dfrac{v + |u|^2 v - (u \cdot v)u}{\sqrt{1 + |u|^2 - |v|^2 - |v|^2|u|^2 + (u \cdot v)^2}}\right)_t - \left(\dfrac{u - |v|^2 u + (u \cdot v)v}{\sqrt{1 + |u|^2 - |v|^2 - |v|^2|u|^2 + (u \cdot v)^2}}\right)_x = 0. \end{cases} \tag{1.26}$$

The initial condition (1.23) together with the boundary condition (1.24) then become

$$t = 0: \quad \big(u(0, x), v(0, x)\big) = U_0(x) = \big(f'(x), g(x)\big), \quad x \geqslant 0, \tag{1.27}$$

$$x = 0: \quad u = 0, \quad t \geqslant 0. \tag{1.28}$$

The system has two n-constant multiple eigenvalues

$$\lambda_\pm(u, v) = \frac{1}{1 + |u|^2}\big(-(u \cdot v) \pm \sqrt{\Delta(u, v)}\big), \tag{1.29}$$

where $\Delta(u, v) = 1 + |u|^2 - |v|^2 - |v|^2|u|^2 + (u \cdot v)^2 > 0$, i.e., the surface is time-like.

Let

$$\begin{cases} R_i = v_i + \lambda_+ u_i \quad (i = 1, \dots, n), \\ S_i = v_i + \lambda_- u_i \quad (i = 1, \dots, n). \end{cases} \tag{1.30}$$

By direct computation, they satisfy the following system (cf. [23,24]):

$$\begin{cases} \dfrac{\partial \lambda_+}{\partial t} + \lambda_- \dfrac{\partial \lambda_+}{\partial x} = 0, \\ \dfrac{\partial R_i}{\partial t} + \lambda_- \dfrac{\partial R_i}{\partial x} = 0 \qquad (i = 1, \dots, n), \\ \dfrac{\partial \lambda_-}{\partial t} + \lambda_+ \dfrac{\partial \lambda_-}{\partial x} = 0, \\ \dfrac{\partial S_i}{\partial t} + \lambda_+ \dfrac{\partial S_i}{\partial x} = 0 \qquad (i = 1, \dots, n), \end{cases} \tag{1.31}$$

$$t = 0: \quad \lambda_+ = \Lambda_+(x), \quad \lambda_- = \Lambda_-(x), \quad R_i = R_i^0(x), \quad S_i = S_i^0(x) \quad (i = 1, \dots, n),\ x \geqslant 0, \tag{1.32}$$

$$x = 0: \quad \lambda_- = -\lambda_+, \quad S_i = R_i \quad (i = 1, \dots, n),\ t \geqslant 0, \tag{1.33}$$

where

$$\Lambda_\pm(x) = \frac{1}{1 + |f'|^2}\big(-(f' \cdot g) \pm \sqrt{1 + |f'|^2 - |g|^2 - |g|^2|f'|^2 + (f' \cdot g)^2}\big), \tag{1.34}$$

$$R_i^0(x) = g_i(x) + \Lambda_+(x) f_i'(x) \quad (i = 1, \dots, n) \tag{1.35}$$

and

$$S_i^0(x) = g_i(x) + \Lambda_-(x) f_i'(x) \quad (i = 1, \dots, n). \tag{1.36}$$

We suppose that the initial data satisfy

$$\max_{x \in R^+} \Lambda_-(x) \leqslant -c < 0 < c \leqslant \min_{x \in R^+} \Lambda_+(x), \tag{1.37}$$

where c is a positive constant.

We assume that the initial data are of bounded but possibly large total variation

$$TV(\Lambda_\pm) := \int_0^{+\infty} \left|\Lambda_\pm'(x)\right| dx \leqslant N, \qquad TV\big(R_i^0\big) := \int_0^{+\infty} \left|\frac{dR_i^0(x)}{dx}\right| dx \leqslant N, \qquad TV\big(S_i^0\big) := \int_0^{+\infty} \left|\frac{dS_i^0(x)}{dx}\right| dx \leqslant N, \tag{1.38}$$

where N is some positive constants (bounded but possibly large).

Example 2 *(Mixed initial–boundary value problem for the system of the motion of relativistic closed strings in the Minkowski space R^{1+n}).* Recently the Born–Infeld theory has received much attention mainly due to the fact that the Born–Infeld type Lagrangians naturally appear in string theory and relativity theory. This triggers the revival of interests in the original Born–Infeld electromagnetism (cf. Born and Infeld [26]) and the exploration of Born–Infeld gauge theory (cf. Gibbons [27]). From the mathematical point of view, this theory is a nonlinear generalization of the Maxwell theory. Gibbons [27] gave a systematic study of the Born–Infeld theory and obtained exact solutions in numerous situations. In this paper, we consider the system which governs the motion of relativistic closed strings in the Minkowski space R^{1+n}. This system is a system with n nonlinear wave equations of Born–Infeld type. This is an interesting model in the Born–Infeld electromagnetic field. It also arises in some physical context (see [9] and the references cited therein). By introducing Riemann invariants, this system can be written as a quasilinear hyperbolic system in diagonal form, which we study in this paper. The Cauchy problem for the system of the motion of relativistic closed strings in the Minkowski space R^{1+n} was studied by Kong, Zhang and Zhou in [9]. They give the necessary and sufficient condition on the global existence of classical solutions of the Cauchy problem for this system. Our goal in this paper is to prove the global existence and uniqueness of classical solutions to the mixed initial–boundary value problem for the system of the motion of relativistic closed strings in the Minkowski space R^{1+n} with large BV data, the specific physical situations for this problem will be discussed in detail in Section 3.

2. Proof of Theorem 1.1

By the local existence and uniqueness of C^1 solution for quasilinear hyperbolic systems [13], there exists $T_0 > 0$ such that the mixed initial–boundary value problem (1.1) and (1.7)–(1.8) admits a unique C^1 solution $u = u(t,x)$ on the domain

$$D(T_0) \overset{def}{=} \{(t,x) \mid 0 \leqslant t \leqslant T_0,\ x \geqslant 0\}. \tag{2.1}$$

Thus, in order to prove Theorem 1.1 it suffices to establish a uniform a priori estimate for the C^0 norm of u and u_x on any given domain of existence of the C^1 solution $u = u(t,x)$.

For any fixed $T > 0$, we introduce

$$w_i(t,x) = \frac{\partial u_i(t,x)}{\partial x} \quad (i = 1, \dots, n), \tag{2.2}$$

$$U_\infty(T) = \sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbf{R}^+} |u(t,x)|, \tag{2.3}$$

$$W_\infty(T) = \sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbf{R}^+} |w(t,x)|, \tag{2.4}$$

$$W_1(T) = \sup_{0 \leqslant t \leqslant T} \int_0^{+\infty} |w(t,x)|\,dx, \tag{2.5}$$

$$\widetilde{W}_1(T) = \max_{i=1,\dots,n} \max_{j \neq i} \sup_{C_j} \int_{C_j} |w_i(t,x)|\,dt, \tag{2.6}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}^n$, $w = (w_1, \dots, w_n)^T$ in which w_i is defined by (2.2), while C_j stands for any given jth characteristic on the domain $[0,T] \times \mathbf{R}^+$.

Lemma 2.1. *Under the assumptions of Theorem* 1.1*, on any given domain of existence* $\{(t,x) \mid 0 \leqslant t \leqslant T,\ x \geqslant 0\}$ *of the* C^1 *solution* $u = u(t,x)$ *to the mixed initial–boundary value problem* (1.1) *and* (1.7)–(1.8)*, there exists a positive constant* K *depending only on* δ*,* c *and* M *such that the following hold*:

$$W_1(T), \widetilde{W}_1(T) \leqslant KN, \tag{2.7}$$

$$W_\infty(T) \leqslant KMe^{KN} \tag{2.8}$$

and

$$U_\infty(T) \leqslant KM. \tag{2.9}$$

Proof. We first estimate $U_\infty(T)$.

(i) For $i = 1, \dots, m$, passing through any fixed point (t,x) in the domain $\{(t,x) \mid 0 \leqslant t \leqslant T,\ x \geqslant 0\}$, we draw the ith backward characteristic C_i which intersects the x-axis at a point $(0,\alpha)$. Noting system (1.1), it is easy to see that $u_i(t,x)$ is constant $\varphi_i(\alpha)$ on this characteristic. Therefore, we have

$$|u_i(t,x)| \leqslant \sup_{\alpha \in \mathbf{R}^+} |\varphi(\alpha)| \leqslant M. \tag{2.10}$$

(ii) For $i = m+1, \ldots, n$, for any fixed point (t, x) in the domain $\{(t, x) \mid 0 \leqslant t \leqslant T,\ x \geqslant 0\}$, we draw the ith backward characteristic C_i: $x = x_i(s; t, x)$ passing through this point. Here, there are only two possibilities:

(a) The ith backward characteristic C_i intersects the x-axis at a point $(0, \alpha)$. Since $u_i(t, x)$ is constant $\varphi_i(\alpha)$ on this characteristic, it follows that

$$\left|u_i(t, x)\right| \leqslant \sup_{\alpha \in \mathbf{R}^+} \left|\varphi(\alpha)\right| \leqslant M. \tag{2.11}$$

(b) The ith backward characteristic C_i intersects the t-axis at a point $(t_0, 0)$. Since $u_i(t, x)$ is constant on this characteristic, it follows that

$$u_i(t, x) = u_i(t_0, 0). \tag{2.12}$$

Noting (1.8), by (1.10), it is easy to get

$$u_i(t_0, 0) = \sum_{r=1}^{m} g_{ir}(t_0) u_r(t_0, 0) + h_i(t_0), \tag{2.13}$$

where

$$g_{ir}(t_0) = \int_0^1 \frac{\partial g_i}{\partial u_r}\big(\alpha(t_0), \tau u_1(t_0, 0), \ldots, \tau u_m(t_0, 0)\big)\, d\tau. \tag{2.14}$$

Thus, noting (2.12), we obtain from (2.10) and (2.13) that

$$\left|u_i(t, x)\right| \leqslant c_1 \left\{ \sum_{r=1}^{m} \left|u_r(t_0, 0)\right| + \left|h_i(t_0)\right| \right\} \leqslant c_2 M, \tag{2.15}$$

where here and henceforth, c_i $(i = 1, 2, \ldots)$ will denote positive constants depending only on δ, c and M.

Combining (2.10) and (2.11), (2.15), we have

$$U_\infty(T) \leqslant c_3 M. \tag{2.16}$$

We next estimate $\widetilde{W}_1(T)$.

To estimate $\widetilde{W}_1(T)$, we need to estimate

$$\int_{C_j} \left|w_i(t, x)\right| dt,$$

where C_j stands for any given jth characteristic on the domain $[0, T] \times \mathbf{R}^+$. Without loss of generality, we assume that C_j intersects the x-axis with point $A(0, \alpha)$, and intersects the line $t = T$ with point B.

(i) For $i = 1, \ldots, m$, passing through point B, we draw the ith backward characteristic C_i which intersects the x-axis at a point $C(0, \beta)$. For fixing the idea, suppose that $\alpha < \beta$.

Differentiating system (1.1) with respect to x, we get

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u) w_i)}{\partial x} = 0, \tag{2.17}$$

equivalently,

$$d\big(w_i(t, x)\big(dx - \lambda_i(u)\, dt\big)\big) = 0. \tag{2.18}$$

We rewrite (2.18) as

$$d\big(\left|w_i(t, x)\right|\big(dx - \lambda_i(u)\, dt\big)\big) = 0. \tag{2.19}$$

By (2.19), using Stokes' formula on the domain ABC, we have

$$\left| \int_{C_j} \left|w_i(t, x)\right| \big(\lambda_j(u) - \lambda_i(u)\big)\, dt \right| = \int_\alpha^\beta \left|w_i(0, x)\right| dx. \tag{2.20}$$

In the definition of $\widetilde{W}_1(T)$, $j \neq i$, then we have from (1.4) that

$$\left|\lambda_j(u) - \lambda_i(u)\right| \geqslant \delta. \tag{2.21}$$

Thus, it follows that

$$\int_{C_j} \left|w_i(t,x)\right| dt \leqslant \frac{1}{\delta} \int_0^{+\infty} \left|\varphi'(x)\right| dx \leqslant c_4 N. \tag{2.22}$$

(ii) For $i = m+1, \dots, n$, we draw the ith backward characteristic C_i passing through point B. Here, there are only two possibilities:

(a) The ith backward characteristic C_i intersects the t-axis at a point $C(\beta, 0)$. By (2.19), using Stokes' formula on the domain OABC, we have

$$\begin{aligned}\left|\int_{C_j} \left|w_i(t,x)\right| \big(\lambda_j(u) - \lambda_i(u)\big)\, dt\right| &\leqslant \int_0^{\alpha} \left|w_i(0,x)\right| dx + \int_0^{\beta} \left|\lambda_i\big(u(t,0)\big)\right| \left|w_i(t,0)\right| dt \\ &\leqslant \int_0^{+\infty} \left|\varphi'(x)\right| dx + c_5 \int_0^{\beta} \left|w_i(t,0)\right| dt.\end{aligned} \tag{2.23}$$

Then, it follows from (2.21) and (2.23) that

$$\int_{C_j} \left|w_i(t,x)\right| dt \leqslant \frac{1}{\delta} \left\{ \int_0^{+\infty} \left|\varphi'(x)\right| dx + c_5 \int_0^{\beta} \left|w_i(t,0)\right| dt \right\}. \tag{2.24}$$

Differentiating the nonlinear boundary condition (1.8) with respect to t, we get

$$\begin{aligned} x = 0: \quad \frac{\partial u_s}{\partial t} &= \sum_{r=1}^{m} \frac{\partial g_s}{\partial u_r} \big(\alpha(t), u_1, \dots, u_m\big) \frac{\partial u_r}{\partial t} \\ &\quad + \sum_{i=1}^{k} \frac{\partial g_s}{\partial \alpha_i} \big(\alpha(t), u_1, \dots, u_m\big) \alpha_i'(t) + h_s'(t) \quad (s = m+1, \dots, n). \end{aligned} \tag{2.25}$$

Thus, using (1.1) and (1.5), we easily see from (2.25) that

$$x = 0: \quad w_s = \sum_{r=1}^{m} f_{sr}(t,u) w_r + \sum_{j=1}^{k} \overline{f}_{sj}(t,u) \alpha_j'(t) + \widetilde{f}_s(t,u) h_s'(t) \quad (s = m+1, \dots, n), \tag{2.26}$$

where f_{sr}, $\overline{f}_{sj}$ and $\widetilde{f}_s$ are continuous functions of t and u.

Noting (2.16), by (2.26), we have

$$\begin{aligned} \int_0^{\beta} \left|w_i(t,0)\right| dt &= \sum_{r=1}^{m} \int_0^{\beta} \left|f_{ir}(t,u) w_r(t,0)\right| dt + \sum_{j=1}^{k} \int_0^{\beta} \left|\overline{f}_{ij}(t,u) \alpha_j'(t)\right| dt + \int_0^{\beta} \left|\widetilde{f}_i(t,u) h_i'(t)\right| dt \\ &\leqslant c_6 \left\{ \sum_{r=1}^{m} \int_0^{\beta} \left|w_r(t,0)\right| dt + \int_0^{+\infty} \big(\left|\alpha'(t)\right| + \left|h'(t)\right|\big)\, dt \right\}. \end{aligned} \tag{2.27}$$

Then, passing through $C(\beta, 0)$, we draw the rth characteristic C_r ($r \in \{1, \dots, m\}$) which intersects the x-axis at point $D(0, x_D)$. By (2.19), using Stokes' formula on the domain COD, we have

$$\left|\int_0^{\beta} \left|w_r(t,0)\right| \big(-\lambda_r(u)\, dt\big)\right| \leqslant \int_0^{x_D} \left|w_r(0,x)\right| dx \leqslant \int_0^{+\infty} \left|\varphi'(x)\right| dx. \tag{2.28}$$

Noting (1.5), we have

$$\left|\lambda_r(u)\right| \geqslant c. \tag{2.29}$$

Thus, it follows that

$$\int_0^{\beta} \left|w_r(t,0)\right| dt \leqslant \frac{1}{c} \int_0^{+\infty} \left|\varphi'(x)\right| dx. \tag{2.30}$$

Then

$$\int_{C_j} \left|w_i(t,x)\right| dt \leqslant c_7 \left\{ \int_0^{+\infty} \left|\varphi'(x)\right| dx + \int_0^{+\infty} \left(\left|\alpha'(t)\right| + \left|h'(t)\right|\right) dt \right\}. \tag{2.31}$$

(b) The ith backward characteristic C_i intersects the x-axis at a point $C(0,\beta)$. By exploiting the same arguments as in (i), we can deduce that

$$\int_{C_j} \left|w_i(t,x)\right| dt \leqslant \frac{1}{\delta} \int_0^{+\infty} \left|\varphi'(x)\right| dx. \tag{2.32}$$

Combining (2.22) and (2.31), (2.32), we have

$$\widetilde{W}_1(T) \leqslant c_8 \left\{ \int_0^{+\infty} \left|\varphi'(x)\right| dx + \int_0^{+\infty} \left(\left|\alpha'(t)\right| + \left|h'(t)\right|\right) dt \right\} \leqslant c_9 N. \tag{2.33}$$

We next estimate $W_\infty(T)$.

(i) For $i = 1, \ldots, m$, for any fixed point (t,x) in the domain $\{(t,x) \mid 0 \leqslant t \leqslant T,\ x \geqslant 0\}$, let $\xi = x_i(s,\alpha)$ be the ith characteristic C_i passing through point (t,x) and intersecting the x-axis at a point $(0,\alpha)$. Then

$$\frac{dx_i(s,\alpha)}{ds} = \lambda_i\big(u\big(s, x_i(s,\alpha)\big)\big), \qquad x_i(t,\alpha) = x \quad \text{and} \quad x_i(0,\alpha) = \alpha. \tag{2.34}$$

Noting (1.2), we rewrite (2.17) as

$$\frac{\partial w_i}{\partial t} + \lambda_i(u) \frac{\partial w_i}{\partial x} = -\sum_{l \neq i} \frac{\partial \lambda_i(u)}{\partial u_l} w_l w_i. \tag{2.35}$$

From (2.34)–(2.35), it is easy to see that along the ith characteristic $\xi = x_i(s,\alpha)$ we have

$$w_i\big(t, x_i(t,\alpha)\big) = w_i(0,\alpha) \exp\left\{ \int_{C_i} \left(-\sum_{l \neq i} \frac{\partial \lambda_i(u)}{\partial u_l} \right) w_l \, dt \right\}. \tag{2.36}$$

Thus, noting (2.16) and (2.33), we get

$$\left|w_i(t,x)\right| \leqslant \left|\varphi'(\alpha)\right| \exp\left\{ c_{10} \int_{C_i} \sum_{l \neq i} |w_l| \, dt \right\} \leqslant M \exp\big\{c_{11} \widetilde{W}_1(T)\big\} \leqslant M e^{c_{12} N}. \tag{2.37}$$

(ii) For $i = m+1, \ldots, n$, for any fixed point (t,x) in the domain $\{(t,x) \mid 0 \leqslant t \leqslant T,\ x \geqslant 0\}$, we draw the ith backward characteristic C_i passing through this point. Here, there are only two possibilities:

(a) The ith backward characteristic C_i intersects the x-axis at a point $(0,\alpha)$. By exploiting the same arguments as in (i), we can deduce that

$$\left|w_i(t,x)\right| \leqslant M e^{c_{13} N}. \tag{2.38}$$

(b) The ith backward characteristic C_i intersects the t-axis at a point $(t_0, 0)$. By integrating (2.35) along C_i from t_0 to t, we obtain

$$w_i(t,x) = w_i(t_0, 0) \exp\left\{ \int_{C_i} \left(-\sum_{l \neq i} \frac{\partial \lambda_i(u)}{\partial u_l} \right) w_l \, dt \right\}. \tag{2.39}$$

By (2.26), we have

$$w_i(t_0, 0) = \sum_{r=1}^{m} f_{ir}\big(t_0, u(t_0,0)\big) w_r(t_0, 0) + \sum_{j=1}^{k} \overline{f}_{ij}\big(t_0, u(t_0,0)\big) \alpha'_j(t_0) + \widetilde{f}_i\big(t_0, u(t_0,0)\big) h'_i(t_0). \tag{2.40}$$

Then, noting (1.9), (2.16) and (2.37), it follows from (2.40) that

$$\left|w_i(t_0,0)\right| \leqslant c_{14} \big\{ M e^{c_{12} N} + M \big\} \leqslant c_{15} M e^{c_{12} N}. \tag{2.41}$$

Thus, it follows from (2.39) that

$$\left|w_i(t,x)\right| \leqslant c_{15} M e^{c_{12} N} \exp\left\{ c_{16} \int_{C_i} \sum_{l \neq i} |w_l| \, dt \right\} \leqslant c_{15} M e^{c_{12} N} \exp\big\{c_{17} \widetilde{W}_1(T)\big\} \leqslant c_{15} M e^{c_{18} N}. \tag{2.42}$$

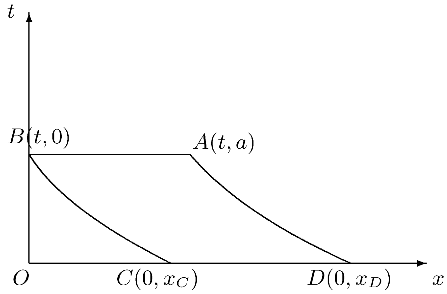

Fig. 1.

Combining (2.37) with (2.38) and (2.42), we obtain

$$W_\infty(T) \leqslant c_{19} M e^{c_{20} N}. \tag{2.43}$$

We finally estimate $W_1(T)$.

To estimate $\int_0^{+\infty} |w_i(t,x)|\,dx$, we need only to estimate

$$\int_0^a \left|w_i(t,x)\right| dx \tag{2.44}$$

for any given $a > 0$ and then let $a \to +\infty$.

(i) For $i = 1, \dots, m$, for any given t with $0 \leqslant t \leqslant T$, passing through point $A(t,a)$ (respectively $B(t,0)$), we draw the ith backward characteristic which intersects the x-axis at a point $D(0, x_D)$ (respectively $C(0, x_C)$), see Fig. 1.

By (2.19), using Stokes' formula on the domain ABCD, we have

$$\int_{BA} \left|w_i(t,x)\right| dx \leqslant \int_{x_C}^{x_D} \left|w_i(0,x)\right| dx \leqslant \int_0^{+\infty} \left|w_i(0,x)\right| dx. \tag{2.45}$$

Thus, we get

$$\int_0^a \left|w_i(t,x)\right| dx \leqslant \int_0^{+\infty} \left|\varphi'(x)\right| dx. \tag{2.46}$$

Letting $a \to +\infty$, we have

$$\int_0^{+\infty} \left|w_i(t,x)\right| dx \leqslant \int_0^{+\infty} \left|\varphi'(x)\right| dx. \tag{2.47}$$

(ii) For $i = m+1, \dots, n$, let $x = x_i(t,0)$ $(0 \leqslant t \leqslant T)$ be the ith forward characteristic passing through origin $O(0,0)$. Then, passing through the point $A(t,a)$ $(a > x_i(t,0))$, we draw the ith backward characteristic which intersects the x-axis at a point $C(0, x_C)$. Let B be the point $(t,0)$. By (2.19), similar to (2.45), using Stokes' formula on the domain ABOC, we have

$$\int_{BA} \left|w_i(t,x)\right| dx \leqslant \int_0^{x_C} \left|w_i(0,x)\right| dx + \int_0^t \lambda_i\big(u(t,0)\big)\left|w_i(t,0)\right| dt. \tag{2.48}$$

Thus, noting (2.16), it follows from (2.48) that

$$\int_0^a \left|w_i(t,x)\right| dx \leqslant \int_0^{+\infty} \left|\varphi'(x)\right| dx + c_{21} \int_0^T \left|w_i(t,0)\right| dt. \tag{2.49}$$

Noting (2.16), by (2.26), we have

$$\begin{aligned}\int_0^T \left|w_i(t,0)\right| dt &= \sum_{r=1}^m \int_0^T \left|f_{ir}(t,u) w_r(t,0)\right| dt + \sum_{j=1}^k \int_0^T \left|\overline{f}_{ij}(t,u)\alpha_j'(t)\right| dt + \int_0^T \left|\widetilde{f}_i(t,u) h_i'(t)\right| dt \\ &\leqslant c_{22}\left\{\sum_{r=1}^m \int_0^T \left|w_r(t,0)\right| dt + \int_0^{+\infty} \left(\left|\alpha'(t)\right| + \left|h'(t)\right|\right) dt\right\}.\end{aligned} \tag{2.50}$$

Then, passing through $D(T,0)$, we draw the rth characteristic $C_r(r\in\{1,\dots,m\})$ which intersects the x-axis at point $E(0,x_E)$. By (2.19), using Stokes' formula on the domain DOE, we have

$$\left|\int_0^T \left|w_r(t,0)\right|\left(-\lambda_r(u)\,dt\right)\right| \leqslant \int_0^{x_E} \left|w_r(0,x)\right|dx \leqslant \int_0^{+\infty} \left|\varphi'(x)\right|dx. \tag{2.51}$$

Noting (1.5), we have

$$\left|\lambda_r(u)\right| \geqslant c. \tag{2.52}$$

Therefore, it follows that

$$\int_0^T \left|w_r(t,0)\right|dt \leqslant \frac{1}{c}\int_0^{+\infty} \left|\varphi'(x)\right|dx. \tag{2.53}$$

Then, noting (2.49)–(2.50), we have

$$\int_0^a \left|w_i(t,x)\right|dx \leqslant c_{23}\left\{\int_0^{+\infty} \left|\varphi'(x)\right|dx + \int_0^{+\infty} \left(\left|\alpha'(t)\right| + \left|h'(t)\right|\right)dt\right\}. \tag{2.54}$$

Letting $a \to +\infty$, we obtain

$$\int_0^{+\infty} \left|w_i(t,x)\right|dx \leqslant c_{23}\left\{\int_0^{+\infty} \left|\varphi'(x)\right|dx + \int_0^{+\infty} \left(\left|\alpha'(t)\right| + \left|h'(t)\right|\right)dt\right\}. \tag{2.55}$$

Combining (2.47) and (2.55), we have

$$W_1(T) \leqslant c_{24}\left\{\int_0^{+\infty} \left|\varphi'(x)\right|dx + \int_0^{+\infty} \left(\left|\alpha'(t)\right| + \left|h'(t)\right|\right)dt\right\} \leqslant c_{25}N. \tag{2.56}$$

Taking K suitably large and noting (2.16), (2.33), (2.43) and (2.56), we obtain (2.7)–(2.9) immediately. Thus, the proof of Lemma 2.1 is finished. □

Proof of Theorem 1.1. The conclusion of Theorem 1.1 follows from Lemma 2.1 immediately. Thus, the proof of Theorem 1.1 is finished. □

3. An application of Theorem 1.1

In this section, we use the conclusion of Theorem 1.1 to consider the mixed initial–boundary value problem for the system of the motion of relativistic closed strings in the Minkowski space R^{1+n}. Recall Kong et al.'s work [9] at first. We denote by $X=(t,x_1,\dots,x_n)$ points in the $(1+n)$-dimensional Minkowski space R^{1+n}. Then the scalar product of two vectors X and $Y=(\widetilde{t},y_1,\dots,y_n)$ in R^{1+n} is defined by

$$X\cdot Y=\sum_{i=1}^n x_i y_i - t\widetilde{t}, \tag{3.1}$$

in particular,

$$X^2=\sum_{i=1}^n x_i^2 - t^2. \tag{3.2}$$

The Lorentzian metric of R^{1+n} can be written as

$$ds^2=\sum_{i=1}^n dx_i^2 - dt^2. \tag{3.3}$$

To describe the motion of a relativistic closed string in the Minkowski space R^{1+n}, we consider the local equation of an extremal time-like surface S in R^{1+n} taking the following parameter form in a suitable coordinate system (cf. [9]):

$$x_i = x_i(t,\theta) \quad (i=1,\dots,n). \tag{3.4}$$

Then, in the surface coordinates t and θ, the Lorentzian metric (3.3) is expressed as

$$ds^2 = (dt, d\theta)M(dt, d\theta)^T, \tag{3.5}$$

where,

$$M = \begin{pmatrix} |x_t|^2 - 1 & \langle x_t, x_\theta\rangle \\ \langle x_t, x_\theta\rangle & |x_\theta|^2 \end{pmatrix}, \tag{3.6}$$

in which $x = (x_1, \dots, x_n)^T$ and

$$\langle x_t, x_\theta\rangle = \sum_{i=1}^n x_{i,t}x_{i,\theta}, \qquad |x_t|^2 = \langle x_t, x_t\rangle \quad \text{and} \quad |x_\theta|^2 = \langle x_\theta, x_\theta\rangle. \tag{3.7}$$

Since the surface S is C^2 and time-like, i.e.,

$$\det M < 0, \tag{3.8}$$

equivalently,

$$\langle x_t, x_\theta\rangle^2 - \big(|x_t|^2 - 1\big)|x_\theta|^2 > 0, \tag{3.9}$$

it follows that the area element of the surface S is

$$dA = \sqrt{\langle x_t, x_\theta\rangle^2 - \big(|x_t|^2 - 1\big)|x_\theta|^2}\, dt\, d\theta. \tag{3.10}$$

The surface S is called to be extremal surface, if $x = x(t,\theta)$ is the critical point of the area functional

$$I = \iint \sqrt{\langle x_t, x_\theta\rangle^2 - \big(|x_t|^2 - 1\big)|x_\theta|^2}\, dt\, d\theta. \tag{3.11}$$

The corresponding Euler–Lagrange equation is (cf. [9])

$$\left(\frac{|x_\theta|^2 x_t - \langle x_t, x_\theta\rangle x_\theta}{\sqrt{\langle x_t, x_\theta\rangle^2 - (|x_t|^2 - 1)|x_\theta|^2}}\right)_t - \left(\frac{\langle x_t, x_\theta\rangle x_t - (|x_t|^2 - 1)x_\theta}{\sqrt{\langle x_t, x_\theta\rangle^2 - (|x_t|^2 - 1)|x_\theta|^2}}\right)_\theta = 0. \tag{3.12}$$

By computation, it follows from (3.12) that

$$|x_\theta|^2 x_{tt} - 2\langle x_t, x_\theta\rangle x_{t\theta} + \big(|x_t|^2 - 1\big)x_{\theta\theta} = 0. \tag{3.13}$$

Remark 3.1. Taking $\theta = x_1$ and $n = 2$, we observe that Eq. (3.13) is just the classical Born–Infeld equation. Therefore, in this sense, Eq. (3.13) is called the generalized Born–Infeld equation.

Let

$$u = x_t, \qquad v = x_\theta, \tag{3.14}$$

where $u = (u_1, \dots, u_n)^T$ and $v = (v_1, \dots, v_n)^T$. Then (3.12) can be equivalently rewritten as

$$\begin{cases} v_t - u_\theta = 0, \\ \left(\dfrac{|v|^2 u - \langle u, v\rangle v}{\sqrt{\langle u, v\rangle^2 - (|u|^2 - 1)|v|^2}}\right)_t - \left(\dfrac{\langle u, v\rangle u - (|u|^2 - 1)v}{\sqrt{\langle u, v\rangle^2 - (|u|^2 - 1)|v|^2}}\right)_\theta = 0. \end{cases} \tag{3.15}$$

We consider the mixed initial–boundary value problem for system (3.15) with the initial condition

$$t = 0: \quad u = u_0(\theta), \quad v = \widetilde{v}_0 + v_0(\theta) \quad (\theta \geqslant 0) \tag{3.16}$$

and the boundary condition

$$\theta = 0: \quad u = 0 \quad (t \geqslant 0). \tag{3.17}$$

Here, $\widetilde{v}_0 = (\widetilde{v}_1^0, \dots, \widetilde{v}_n^0)^T$ is a constant vector with $|\widetilde{v}_0| = \sqrt{(\widetilde{v}_1^0)^2 + \dots + (\widetilde{v}_n^0)^2} > 0$, $u_0(\cdot) = (u_1^0(\cdot), \dots, u_n^0(\cdot))^T$ and $v_0(\cdot) = (v_1^0(\cdot), \dots, v_n^0(\cdot))^T \in C^1$ with bounded C^1 norm, such that

$$\big\|u_0(\theta)\big\|_{C^0}, \big\|v_0(\theta)\big\|_{C^0}, \big\|u_0'(\theta)\big\|_{C^0}, \big\|v_0'(\theta)\big\|_{C^0} \leqslant M, \tag{3.18}$$

for some positive constant M (bounded but possibly large). Also, we assume that the conditions of C^1 compatibility are satisfied at the point $(0, 0)$.

Let

$$U=\begin{pmatrix} u\\ v\end{pmatrix}. \tag{3.19}$$

Then, we can rewrite system (3.15) as

$$U_t+A(U)U_\theta=0, \tag{3.20}$$

where

$$A(U)=\begin{bmatrix} -\frac{2\langle u,v\rangle}{|v|^2}I_{n\times n} & \frac{|u|^2-1}{|v|^2}I_{n\times n}\\ -I_{n\times n} & 0\end{bmatrix}. \tag{3.21}$$

It is easy to see that in a neighborhood of $U_0=\binom{0}{\tilde{v}_0}$, (3.15) is a hyperbolic system with the following real eigenvalues:

$$\lambda_1(U)\equiv\cdots\equiv\lambda_n(U)=\lambda_-<0<\lambda_{n+1}(U)\equiv\cdots\equiv\lambda_{2n}(U)=\lambda_+, \tag{3.22}$$

where

$$\lambda_\pm=\frac{-\langle u,v\rangle\pm\sqrt{\langle u,v\rangle^2-(|u|^2-1)|v|^2}}{|v|^2}. \tag{3.23}$$

The corresponding left and right eigenvectors are

$$l_i(U)=(e_i,\lambda_+e_i)\quad(i=1,\dots,n),\qquad l_i(U)=(e_{i-n},\lambda_-e_{i-n})\quad(i=n+1,\dots,2n) \tag{3.24}$$

and

$$r_i(U)=(-\lambda_-e_i,e_i)^T\quad(i=1,\dots,n),\qquad r_i(U)=(-\lambda_+e_{i-n},e_{i-n})^T\quad(i=n+1,\dots,2n) \tag{3.25}$$

respectively, where

$$e_i=(0,\dots,0,\overset{(i)}{1},0,\dots,0)\quad(i=1,\dots,n). \tag{3.26}$$

When $n=1$, (3.15) is a strictly hyperbolic system; while, when $n\geqslant 2$, (3.15) is a non-strictly hyperbolic system with characteristics with constant multiplicity. It is easy to see that all characteristic fields are linearly degenerate in the sense of Lax, i.e.,

$$\nabla\lambda_i(U)r_i(U)\equiv 0\quad(i=1,\dots,2n). \tag{3.27}$$

The fact that system (3.15) is linearly degenerate follows from the constant multiplicity of the eigenspeeds (3.22), through a general theorem by Boillat (see Serre [18, vol. I]).

Let

$$\begin{cases} R_i=u_i+\lambda_+v_i\quad(i=1,\dots,n),\\ S_i=u_i+\lambda_-v_i\quad(i=1,\dots,n).\end{cases} \tag{3.28}$$

Since

$$\Delta(u,v)\overset{\triangle}{=}\langle u,v\rangle^2-\big(|u|^2-1\big)|v|^2>0, \tag{3.29}$$

by a direct computation, (3.15) implies that

$$\begin{cases} \dfrac{\partial\lambda_+}{\partial t}+\lambda_-\dfrac{\partial\lambda_+}{\partial\theta}=0,\\ \dfrac{\partial R_i}{\partial t}+\lambda_-\dfrac{\partial R_i}{\partial\theta}=0\quad(i=1,\dots,n),\\ \dfrac{\partial\lambda_-}{\partial t}+\lambda_+\dfrac{\partial\lambda_-}{\partial\theta}=0,\\ \dfrac{\partial S_i}{\partial t}+\lambda_+\dfrac{\partial S_i}{\partial\theta}=0\quad(i=1,\dots,n).\end{cases} \tag{3.30}$$

The initial condition (3.16) together with the boundary condition (3.17) then can be rewritten as

$$t=0:\quad \lambda_+(0,\theta)=\Lambda_+(\theta),\quad \lambda_-(0,\theta)=\Lambda_-(\theta),\quad R_i(0,\theta)=R_i^0(\theta),\quad S_i(0,\theta)=S_i^0(\theta)$$
$$(i=1,\dots,n),\ \theta\geqslant 0, \tag{3.31}$$

$$\theta=0:\quad \lambda_-=-\lambda_+,\quad S_i=-R_i\quad(i=1,\dots,n),\ t\geqslant 0, \tag{3.32}$$

where

$$\Lambda_{\pm}(\theta)=\frac{1}{|\widetilde{v}_0+v_0(\theta)|^2}\Bigl(-\bigl\langle u_0(\theta),\widetilde{v}_0+v_0(\theta)\bigr\rangle\pm\sqrt{\bigl\langle u_0(\theta),\widetilde{v}_0+v_0(\theta)\bigr\rangle^2-\bigl(\bigl|u_0(\theta)\bigr|^2-1\bigr)\bigl|\widetilde{v}_0+v_0(\theta)\bigr|^2}\Bigr), \tag{3.33}$$

$$R_i^0(\theta)=u_i^0(\theta)+\Lambda_+(\theta)\bigl(\widetilde{v}_i^0+v_i^0(\theta)\bigr)\quad(i=1,\dots,n) \tag{3.34}$$

and

$$S_i^0(\theta)=u_i^0(\theta)+\Lambda_-(\theta)\bigl(\widetilde{v}_i^0+v_i^0(\theta)\bigr)\quad(i=1,\dots,n). \tag{3.35}$$

Then, by Theorem 1.1, we get the following global existence result.

Theorem 3.1 *(Global existence). Suppose that u_0, v_0 are all C^1 functions with respect to their arguments satisfying the conditions of C^1 compatibility at the point $(0,0)$. If (3.18) holds together with*

$$TV(\Lambda_{\pm}):=\int_0^{+\infty}\bigl|\Lambda'_{\pm}(\theta)\bigr|\,d\theta\leqslant N,\qquad TV\bigl(R_i^0\bigr):=\int_0^{+\infty}\left|\frac{dR_i^0(\theta)}{d\theta}\right|d\theta\leqslant N,\qquad TV\bigl(S_i^0\bigr):=\int_0^{+\infty}\left|\frac{dS_i^0(\theta)}{d\theta}\right|d\theta\leqslant N, \tag{3.36}$$

where M and N are some positive constants (bounded but possibly large). Then the mixed initial–boundary value problem (3.15)–(3.17) admits a unique global C^1 solution $U=U(t,\theta)$, defined in the half space $\{(t,\theta)\mid t\geqslant 0,\ \theta\geqslant 0\}$, with bounded total variation in θ uniformly for all $t\geqslant 0$.

4. Concluding summary

(1) In this paper we proved the global existence and uniqueness of classical solutions to the mixed initial–boundary value problem with large BV data for linearly degenerate quasilinear hyperbolic systems of diagonal form. Then we applied the result to obtain the global existence of classical solutions to the mixed initial–boundary value problem with large BV data for the system describing the motion of relativistic closed strings in the Minkowski space R^{1+n}. The similar conclusion can be obtained for the initial–boundary value problem with homogeneous Neumann boundary condition with large BV data for the equation of time-like extremal surfaces in Minkowski space $R^{1+(1+n)}$. Compared with previous literature, the main novelty of the present result stems from the fact that here the C^1 norm and the BV norm of the initial and boundary data are bounded but possibly large, there are no various smallness assumptions on the initial and boundary data by Li et al. [11,12], Shao [19–21] etc. However, we left open the question of the global existence and uniqueness of classical solutions for general linearly degenerate quasilinear hyperbolic systems with large BV data. Since the special forms of the systems under our consideration play an important role in the corresponding analysis.

(2) It is also very important and interesting to study the case where some of the characteristic fields are genuinely nonlinear. Donadello and Marson's work [25] shows that the initial–boundary value problem remains well posed within the class of small BV solutions, even when some of the characteristic fields are for genuinely nonlinear. Therefore this problem is worthy of studying in the future.

Acknowledgments

The author would like to thank the referees for their pertinent comments and valuable suggestions. The author thanks Prof. Ta-tsien Li for his valuable lectures and guidance in Fuzhou in December, 2003. The author also wishes to thank Prof. Jiaxing Hong for his guidance and constant encouragement.

References

[1] D. Amadori, W. Shen, The slow erosion limit in a model of granular flow, preprint available at http://www.math.ntnu.no/conservation/2009/003.html.
[2] D. Amadori, W. Shen, Global existence of large BV solutions in a model of granular flow, preprint available at http://www.math.ntnu.no/conservation/2008/023.html.
[3] A. Bressan, Contractive metrics for nonlinear hyperbolic systems, Indiana Univ. Math. J. 37 (1988) 409–421.
[4] A. Bressan, Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem, Oxford University Press, 2000.
[5] W.-R. Dai, D.-X. Kong, Global existence and asymptotic behavior of classical solutions of quasilinear hyperbolic systems with linearly degenerate characteristic fields, J. Differential Equations 235 (2007) 127–165.
[6] J. Glimm, P.D. Lax, Decay of solutions of systems of nonlinear hyperbolic conservation laws, Mem. Amer. Math. Soc. 101 (1970).
[7] D.-X. Kong, Cauchy Problem for Quasilinear Hyperbolic Systems, MSJ Mem., vol. 6, Mathematical Society of Japan, Tokyo, 2000.
[8] D.-X. Kong, Life-span of classical solutions to quasilinear hyperbolic systems with slow decay initial data, Chinese Ann. Math. Ser. B 21 (2000) 413–440.
[9] D.-X. Kong, Q. Zhang, Q. Zhou, The dynamics of relativistic strings moving in the Minkowski space R^{1+n}, Comm. Math. Phys. 269 (2007) 153–174.
[10] T.-T. Li, Y.-J. Peng, Cauchy problem for weakly linearly degenerate hyperbolic systems in diagonal form, Nonlinear Anal. 55 (2003) 937–949.
[11] T.-T. Li, Y.-J. Peng, The mixed initial–boundary value problem for reducible quasilinear hyperbolic systems with linearly degenerate characteristics, Nonlinear Anal. 52 (2003) 573–583.
[12] T.-T. Li, L.-B. Wang, Global classical solutions to a kind of mixed initial–boundary value problem for quasilinear hyperbolic systems, Discrete Contin. Dyn. Syst. 12 (2005) 59–78.
[13] T.-T. Li, W.-C. Yu, Boundary Value Problems for Quasilinear Hyperbolic Systems, Duke University Press, 1985.

[14] T.-T. Li, Y. Zhou, D.-X. Kong, Weak linear degeneracy and global classical solutions for general quasilinear hyperbolic systems, Comm. Partial Differential Equations 19 (1994) 1263–1317.
[15] T.-T. Li, Y. Zhou, D.-X. Kong, Global classical solutions for general quasilinear hyperbolic systems with decay initial data, Nonlinear Anal. 28 (1997) 1299–1332.
[16] J. Liu, Y. Zhou, Asymptotic behaviour of global classical solutions of diagonalizable quasilinear hyperbolic systems, Math. Methods Appl. Sci. 30 (2007) 479–500.
[17] T. Nishida, J. Smoller, Solutions in the large for some nonlinear conservation laws, Comm. Pure Appl. Math. 26 (1973) 183–200.
[18] D. Serre, Systems of Conservation Laws I, II, Cambridge University Press, 2000.
[19] Z.-Q. Shao, The mixed initial–boundary value problem for quasilinear hyperbolic systems with linearly degenerate characteristics, Nonlinear Anal. 71 (2009) 1350–1368.
[20] Z.-Q. Shao, Global weakly discontinuous solutions for hyperbolic conservation laws in the presence of a boundary, J. Math. Anal. Appl. 345 (2008) 223–242.
[21] Z.-Q. Shao, Global solution to the generalized Riemann problem in the presence of a boundary and contact discontinuities, J. Elasticity 87 (2007) 277–310.
[22] Y. Zhou, Global classical solutions to quasilinear hyperbolic systems with weak linear degeneracy, Chinese Ann. Math. Ser. B 25 (2004) 37–56.
[23] D.-X. Kong, Q.-Y. Sun, Y. Zhou, The equation for time-like extremal surfaces in Minkowski space R^{2+n}, J. Math. Phys. 47 (2006) 013503-1–013503-16.
[24] J. Liu, Y. Zhou, The initial–boundary value problem on a strip for the equation of time-like extremal surfaces, Discrete Contin. Dyn. Syst. 23 (2009) 381–397.
[25] C. Donadello, A. Marson, Stability of front tracking solutions to the initial and boundary value problem for systems of conservation laws, NoDEA Nonlinear Differential Equations Appl. 14 (2007) 569–592.
[26] M. Born, L. Infeld, Foundations of the new field theory, Proc. Roy. Soc. London A 144 (1934) 425–451.
[27] G.W. Gibbons, Born–Infeld particles and Dirichlet p-branes, Nuclear Phys. B 514 (1998) 603–639.